PHYSICAL REVIEW B VOLUME 1, NUMBER 5 1 MARCH 1970

Critical Behavior of the Four-Dimensional Ising Ferromagnet and the Breakdown of Scaling*

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(Received 9 September 1969)

Experience obtained from an analysis of the high-temperature series expansion of the spin-spin correlation function of the four-dimensional Ising model shows that the apparent failure of scaling in the three-dimensional Ising model cannot easily be attributed to logarithmic factors modifying the simple power-law singularities at the critical point.

THE failure of the scaling relation[1,2]

$$d\nu=2-\alpha \tag{1}$$

between the correlation index ν and the high-temperature specific-heat index α, now seems well established for Ising systems when d, the dimensionality of the system, equals three.[3–5] The scaling relation predicts that $\nu=0.625$, given that the specific-heat index is $\frac{1}{8}$. This estimate for α was initially obtained from series,[6] but has recently been confirmed by measurements at the ordering transition in beta brass.[7] However, both series analysis and scattering experiments[8,9] yield values for ν in the range 0.638–0.647. Since the scaling hypothesis generally has been of such use in the study of continuous phase transitions, it would seem only sensible to question whether the series data and the experiments are yielding the true value of ν. One possibility that has been raised by several authors is that κ, the inverse correlation length, does not behave as ϵ^ν when $\epsilon(=(1-T_c/T))\rightarrow 0$, but instead as $\epsilon^\nu|\ln h\epsilon|^{-x}$, where $\nu=0.625$ and h and x are positive constants.[10,11] Such a logarithmic term could easily result in ν appearing to be about 0.64. Now it is widely expected that the four-dimensional Ising model will have mean-field critical indices, i.e., $\gamma=1$, $\nu=\frac{1}{2}$, $\alpha=0$, but modified by logarithmic terms.[12,13] We therefore generated series expansions for its spin-spin correlation function, $\Gamma(\mathbf{r},v)$. The experience gained from dealing with the conjectured logarithmic terms shows that logarithmic terms are a most unlikely explanation for the failure of (1) for the three-dimensional Ising model, and therefore that the breakdown of scaling is a genuine and fundamental effect.[5]

The high-temperature series expansion of $\Gamma(\mathbf{r},v)$ was found to order v^{10} on the hfcc lattice and to order v^{11} on the hbcc and hsc (hyper simple cubic) lattices, where $v=\tanh(J/k_BT)$. The hfcc lattice (h stands for hyper, fcc denotes the three-dimensional analog) is the lattice whose site labels (i,j,k,l), taken to be integers, are such that $(i+j+k+l)$ is even; the hbcc lattice has site labels which are either all even or all odd; the hsc lattice is the lattice usually referred to as the hypercubic lattice.[14] The method used in obtaining the series has been outlined elsewhere.[4,15] A check on the computer program was the agreement of our susceptibility series on the hsc lattice with some earlier work of Fisher and Gaunt.[14] Tables I and II contain some of the data produced in the calculation.[16]

In order to determine the critical value v_c and the indices ν and γ, it is convenient to form the spherical moment series[3] defined by

$$\mu_t=\sum_{\mathbf{r}} r^t\Gamma(\mathbf{r},v)=\sum_n m_n^{(t)}v^n. \tag{2}$$

The quantities $m_n^{(0)}$ are the coefficients of the usual dimensionless susceptibility series; $m_n^{(2)}$ are the coefficients of the second moment series. Table I contains both these sets of coefficients for the three lattices. It is expected that μ_t will have a leading singularity of the form[3]

$$\mu_t\sim[1-(v/v_c)]^{-\gamma-t\nu}, \quad \text{as } v\rightarrow v_c \tag{3}$$

where γ is the susceptibility index $(t=0)$. Using the customary ratio methods of analysis[17] on the various moment series yields the estimates $v_c(\text{hfcc})=1/21.984$, $v_c(\text{hbcc})=1/14.510$, and $v_c(\text{hsc})=1/6.725$, with an un-

* Work supported in part by the National Science Foundation under Grant No. NSF GP-7765 and by the Advanced Research Projects Agency under Contract No. ARPA SD-131.

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16 To display the entire expansion of the correlation function on all three lattices would take a large number of pages. Similarly it is possible to give only the briefest outline of the analysis. Our aim therefore is to present enough series data so that the reader, if he so wishes, can directly check our most important claims.

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TABLE I. Coefficients for the expansion of the zeroth and second correlation moments in powers of v for the four-dimensional Ising ferromagnet.

	hsc		hbcc		hfcc	
n	$m_n^{(0)}$	$m_n^{(2)}$	$m_n^{(0)}$	$m_n^{(2)}$	$m_n^{(0)}$	$m_n^{(2)}$
1	8	8	16	16	24	24
2	56	128	240	512	552	1 152
3	392	1 416	3 600	11 792	12 504	40 344
4	2 696	13 568	53 200	238 592	280 920	1 237 632
5	18 536	119 912	786 000	4 480 592	6 280 824	35 267 640
6	126 536	1 008 000	11 543 952	80 330 752	139 986 456	958 763 904
7	863 720	8 181 224	169 538 256	1 393 717 968	3 113 177 400	25 226 225 976
8	5 873 768	64 741 376	2 482 215 952	23 613 960 192	69 123 011 832	647 984 522 496
9	39 942 184	502 411 816	36 341 716 432	392 797 482 960	1 532 877 910 008	16 341 539 178 744
10	271 009 112	3 839 881 088	531 081 176 880	6 440 613 392 384	33 960 365 608 008	406 175 271 393 384
11	1 838 725 896	28 985 476 360	7 760 918 381 072	104 370 678 165 520		

certainty in each case of probably no more than two or three in the last decimal place. All the analysis methods mentioned in Ref. 4 were then employed and we found that $\gamma=1.065\pm0.003$, $\nu=0.536\pm0.003$, and $\alpha=-0.12\pm0.03$. These results are independent of the lattice considered, within the error limits quoted.[18] These limits do not of course represent absolute limits, but rather the subjective uncertainty in reading of Neville tables and the like. The regularity of the series can be judged by the sequence of estimates for γ, obtained from the formula

$$\gamma_n-1=n(v_c\rho_n-1)\,, \tag{4}$$

where ρ_n is the ratio of successive terms in the susceptibility series on the hfcc lattice. For $n=5$ to $n=10$, the γ_n are 1.08507, 1.08295, 1.08124, 1.07983, 1.07864, and 1.07762. Allowing for the slight downward trend gives the above result for γ. The value obtained for ν is independent of the moment series from which it was derived, at least for t in the range from -2 to 10. Our series are too short to give accurate results outside this range. The same value for ν of 0.536 is also obtained on analyzing the series for the true correlation length.[3] The constancy of ν derived from different series is a feature of the three-dimensional Ising model.[4] The small discrepancies between the observed values and the mean-field values are of the same magnitude as the difference between the observed value of ν in three dimensions and the scaling prediction.

To show how the presence of logarithmic factors could modify the observed values of the critical indices, it is convenient to proceed as follows. The series expansion of the spin-spin correlation function can be written as

$$\Gamma(\mathbf{r},v)=\textstyle\sum_n q_n(\mathbf{r})(v/v_c)^n\,. \tag{5}$$

If strong scaling holds,[1,3,5] that is, if

$$\Gamma(\mathbf{r},v)=D(\kappa r)/r^{d-2+\eta}\,,\quad \text{with}\quad r\gg b,\ \kappa b\ll 1 \tag{6}$$

but for arbitrary values of the product κr, then $q_n(\mathbf{r})$ will have the form

$$q_n(\mathbf{r})=(c_n/a_n{}^d)F(r/a_n)\,,\quad r\gg b,\ n\gg 1 \tag{7}$$

where $a_n=a_0n^\nu$ and $c_n=c_0n^{\gamma-1}$; b is the lattice spacing. Substituting (7) into (2) yields

$$m_n{}^{(t)}=A_tv_c{}^{-n}n^{\gamma-1+t\nu}\,,\quad n\gg 1 \tag{8}$$

where A_t is a constant. Logarithmic factors can be considered in a fairly general way by having $a_n=a_0n^\nu(\ln an)^x$ and $c_n=c_0n^{\gamma-1}(\ln cn)^y$, where a, x, c, and y are constants. By substituting (7) into (5), it can be seen that these expressions for a_n and c_n give rise to an inverse coherence length which behaves as $\epsilon^\nu|\ln\epsilon|^{-x}$, a susceptibility which varies like $\epsilon^{-\gamma}|\ln\epsilon|^y$, while at $T=T_c$, they give

$$\Gamma(\mathbf{r},v_c)\sim(\ln r)^p/r^{d-2+\eta}\,,\quad r\to\infty \tag{9}$$

with $p=y-x\gamma/\nu$. For

$$r\ll a_n,\quad F(z)\sim z^g\quad\text{and}\quad g=(\gamma+1-d\nu-\alpha)/\nu\,,$$

provided strong scaling holds.[5] Hence,

$$q_n(\mathbf{r})\sim n^{-2+\alpha}r^g(\ln cn)^y(\ln an)^{-dx-xg}\,,\quad n\to\infty\,. \tag{10}$$

In the ratio method[17] of determining indices [see Eq. (4)], the presence of the logarithmic terms results in an effective γ_n, ν_n, and α_n, related to the true γ, ν,

TABLE II. Expansion coefficients of $\Gamma(\mathbf{r},v)$ at the nearest-neighbor site.

n	hsc	hbcc	hfcc
1	1	1	1
2			8
3	6	50	94
4			1 232
5	108	4 264	17 916
6			278 088
7	2 628	474 208	4 536 836
8			76 800 064
9	72 638	61 159 218	1 338 190 014
10			23 861 081 448
11	2 200 108	8 651 484 328	

[18] The work of Ref. 14 gives $v_c(\text{hsc})=1/6.7220\pm0.0015$ and $\gamma=1.094\pm0.0025$. This discrepancy is due to the use of different analysis methods. In our experience, Neville tables give more reliable results than the analysis methods of Ref. 14.

and α by

$$\gamma_n^{\text{eff}}=\gamma+y/\ln cn\,, \tag{11a}$$

$$\nu_n^{\text{eff}}=\nu+x/\ln an\,, \tag{11b}$$

and

$$\alpha_n^{\text{eff}}=\alpha+y/\ln cn-x(d+g)/\ln an\,, \tag{11c}$$

as $n\to\infty$. Since the logarithmic terms vary slowly with n, it will be very difficult to distinguish between their presence and a genuine shift of the indices away from their mean-field values. If we knew a, x, c, and y, it would be possible to produce series free of the influence of the logarithmic terms by forming the coefficients

$$\bar{m}_n^{(t)}=m_n^{(t)}(\ln cn)^{-y}(\ln an)^{-tx} \tag{12a}$$

and

$$\bar{q}_n(\mathbf{r})=q_n(\mathbf{r})(\ln cn)^{-y}(\ln an)^{x(d+g)}\,. \tag{12b}$$

Analysis of these coefficients should yield the "true" indices. In fact, none of the constants are known, so we tried instead to find values of a, c, x, and y which yielded the mean-field values of γ, ν, and α. With $2x=y=1$ (this gives the critical behavior of the four-dimensional spherical model[19]), and $a=c=25\,000$, it was found that $\gamma=0.994\pm0.003$, $\nu=0.504\pm0.003$, and $\alpha=-0.05\pm0.03$ for the hfcc lattice. Allowing c to differ from a resulted in an even closer approach to $\gamma=1$, $\nu=\frac{1}{2}$, and $\alpha=0$. The series for $\bar{m}_n^{(t)}$ and $\bar{q}_n(\mathbf{r})$ with these values for the constants were as regular as the original series—a fact reflected in the error limits—and the values of ν obtained from different moment series were again the same throughout the range $-2<t<10$. Notice that the first estimates of γ^{eff}, ν^{eff}, and α^{eff}, viz., 1.065, 0.536, and -0.12, respectively, roughly satisfy Eqs. (11) with the quoted values of a, c, x, and y and the second estimates used for γ, ν, and α. It seems fair to conclude that the series data is suggestive of logarithmic factors for the four-dimensional Ising model, but that it is unable to distinguish between them and a real departure of the indices from their mean-field values.

However, a stronger statement can be made concerning the possibility of logarithmic terms as an explanation of the failure of the scaling relation (1) for the three-dimensional Ising model. If it is granted that the *present* values for γ and α are the *true* values, then (11) and our experience of the four-dimensional model show that $y/\ln cn$ and $x/\ln an$ must either be zero or very small, so that ν_n^{eff} could hardly differ much from ν. To consider the matter in more detail, we set $y=0$ and $x=\frac{1}{12}$,[20] and varied the constant a until the value 0.625 for ν was obtained on analyzing $\bar{m}_n^{(t)}$. With $a=10$, we found $\gamma=5/4$, $\nu=0.626\pm0.006$, and $\alpha=0.18\pm0.02$. The large error limits on ν are due to the fact that when the $\bar{m}_n^{(t)}$ with $t<0$ are analyzed, rather low values of ν (around 0.621) are obtained. Note that the introduction of logarithmic terms has altered the observed value of α, as predicted by Eq. (11c). It seemed impossible to introduce logarithms to remove the discrepancy in ν without either changing γ or α. As there seems no reason at present to change these indices, we conclude that logarithmic factors cannot explain the breakdown of scaling.

The author thanks D. S. McKenzie for help with the programming, D. Jasnow for pointing out the method of obtaining the series for the true correlation length, and M. Ferer for aiding with the analysis.

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[20] The value of $x=\frac{1}{12}$ with $y=0$ is compatible with a suggestion of Stell (Ref. 11). We set $y=0$ as we presumed that the result 5/4 for γ was more firmly established than the value $\frac{1}{8}$ for α—specific-heat series and experiments being notoriously hard to analyze correctly.